CHAPTER

1

DIFFERENTIATION

Syllabus coverage

TOPIC 3.1: FURTHER DIFFERENTIATION AND APPLICATIONS

Differentiation rules

3.1.7 examine and use the product and quotient rules

3.1.8 examine the notion of composition of functions and use the chain rule for determining the derivatives of composite functions

Mathematics Methods ATAR Course Year 12 syllabus p. 9, © SCSA

Video playlists (6):

1.1 Differentiating simple functions
1.2 The product rule
1.3 The quotient rule
1.4 The chain rule
1.5 Combining the rules
WACE question analysis Differentiation

Worksheets (8):

1.1 Derivatives of polynomials • Rates of change 2 • Instantaneous rates of change • Slopes of curves
1.2 The product rule
1.3 The quotient rule
1.4 The chain rule • Mixed differentiation problems

Nelson MindTap

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1.1 Differentiating simple functions

In Year 11 we learnt that the derivatives of $f(x) = x$, $f(x) = x^2$, $f(x) = x^3$ and $f(x) = x^4$ can be found using **differentiation** by first principles, with the results shown in the table.

$f(x)$	$f'(x)$	$\frac{d}{dx}$
x	$f'(x) = 1$	$\frac{d}{dx}(x) = 1$
x^2	$f'(x) = 2x$	$\frac{d}{dx}(x^2) = 2x$
x^3	$f'(x) = 3x^2$	$\frac{d}{dx}(x^3) = 3x^2$
x^4	$f'(x) = 4x^3$	$\frac{d}{dx}(x^4) = 4x^3$

Video playlist Differentiating simple functions

Worksheets Derivatives of polynomials

Rates of change 2

Instantaneous rates of change

Slopes of curves

We also discovered the pattern in finding the **derivative** of $f(x) = x^n$, where $n = 1, 2, 3 \ldots$ A similar pattern can be found for the derivative of $f(x) = ax^n$.

To differentiate this function, multiply the coefficient a by the power and subtract 1 from the power.

This rule applies to all types of **powers of *x*** (integers, fractions, surds and irrationals).

If $y = f(x)$ then $f'(a)$ is the **instantaneous rate of change** of the function f at a given point $x = a$.

The **tangent** to a curve is a straight line touching the curve at a point.

A **stationary point** is a point on a curve where the gradient is zero (that is, where $f'(x) = 0$).

The derivative of ax^n

If $f(x) = ax^n$, then $f'(x) = anx^{n-1}$.

To differentiate a function with more than one term, use term-by-term differentiation.

For example, to differentiate $f(x) = 2x^2 + 3x - 1$, differentiate each term separately:

$\frac{d}{dx}(2x^2) + \frac{d}{dx}(3x) + \frac{d}{dx}(-1)$, which gives $f'(x) = 4x + 3$.

y' means $\frac{dy}{dx}$.

Exam hack

To help when differentiating, make sure that each term is of the form ax^n.

For example, write $\frac{9}{x^2}$ as $9x^{-2}$ and, hence, $\frac{d}{dx}(9x^{-2}) = -18x^{-3} = -\frac{18}{x^3}$.

WORKED EXAMPLE 1 The derivative of ax^n

Differentiate each function.

a $y = 5x^7$ **b** $f(x) = \dfrac{9}{10x^{\frac{2}{3}}}$ **c** $f(x) = x^2 + 2\sqrt{3x^{\frac{7}{2}}}$

Steps	Working
a **1** Write in the form $y = ax^n$.	$y = 5x^7$
2 Differentiate using $y' = a \times nx^{n-1}$.	$y' = 5 \times 7x^{7-1}$
3 Simplify.	$y' = 35x^6$
b **1** Write in the form $f(x) = ax^n$.	$f(x) = \dfrac{9}{10x^{\frac{2}{3}}} = \dfrac{9}{10}x^{-\frac{2}{3}}$
2 Differentiate using $f'(x) = a \times nx^{n-1}$.	$f'(x) = \dfrac{9}{10} \times \left(-\dfrac{2}{3}\right)x^{-\frac{2}{3}-1} = -\dfrac{3}{5}x^{-\frac{5}{3}} = -\dfrac{3}{5x^{\frac{5}{3}}}$
c **1** Write in the form $f(x) = ax^n$.	$f(x) = x^2 + 2\sqrt{3x^{\frac{7}{2}}} = x^2 + 2\left(3x^{\frac{7}{2}}\right)^{\frac{1}{2}} = x^2 + 2\sqrt{3}x^{\frac{7}{4}}$
2 Differentiate using $f'(x) = a \times nx^{n-1}$.	$f'(x) = 2x + 2\sqrt{3} \times \dfrac{7}{4}x^{\frac{7}{4}-1} = 2x + \dfrac{7\sqrt{3}}{2}x^{\frac{3}{4}}$

WORKED EXAMPLE 2 The derivative at a given point

For each function $f(x)$, calculate $f'(x)$ using the given value of x.

a $f(x) = 3x^4 - 2x^2 + 1, f'(1)$ **b** $f(x) = \dfrac{x^2 - 4}{x + 2}, f'(3)$ **c** $f(x) = \dfrac{3x^{\frac{7}{5}} + 4x^{\frac{12}{5}}}{x^{\frac{2}{5}}}, f'\left(\dfrac{1}{8}\right)$

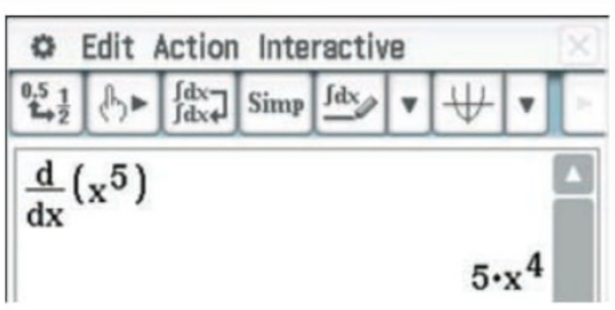
Steps	Working
a 1 Differentiate each term.	$f'(x) = 12x^3 - 4x$
2 Evaluate $f'(1)$.	$f'(1) = 12(1)^3 - 4(1)$ $= 8$
b 1 Factorise and simplify $f(x)$.	$f(x) = \dfrac{x^2 - 4}{x + 2}$ $= \dfrac{(x + 2)(x - 2)}{x + 2}$ $= x - 2$
2 Differentiate each term.	$f'(x) = 1$, for $x \neq -2$
3 Evaluate $f'(x)$.	$f'(3) = 1$
c 1 Simplify $f(x)$.	$f(x) = \dfrac{3x^{\frac{7}{5}} + 4x^{\frac{12}{5}}}{x^{\frac{2}{5}}}$ $= 3x^{\frac{7}{5} - \frac{2}{5}} + 4x^{\frac{12}{5} - \frac{2}{5}}$ $= 3x + 4x^2$
2 Differentiate each term.	$f'(x) = 3 + 8x$
3 Evaluate $f'\left(\dfrac{1}{8}\right)$.	$f'\left(\dfrac{1}{8}\right) = 3 + 8\left(\dfrac{1}{8}\right)$ $= 4$

We can differentiate a function using CAS.

USING CAS 1 Finding the derivative

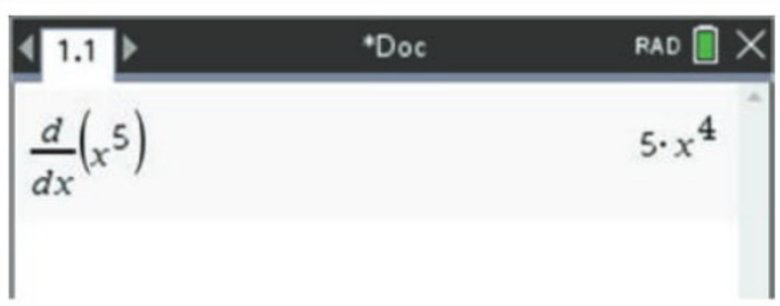
Find the derivative of $f(x) = x^5$.

ClassPad



1 In **Main** enter and highlight the expression x^5.
2 Tap **Interactive** > **Calculation** > **diff.**
3 In the dialogue box, keep the default **Variable: x** and the **Order: 1**.

TI-Nspire



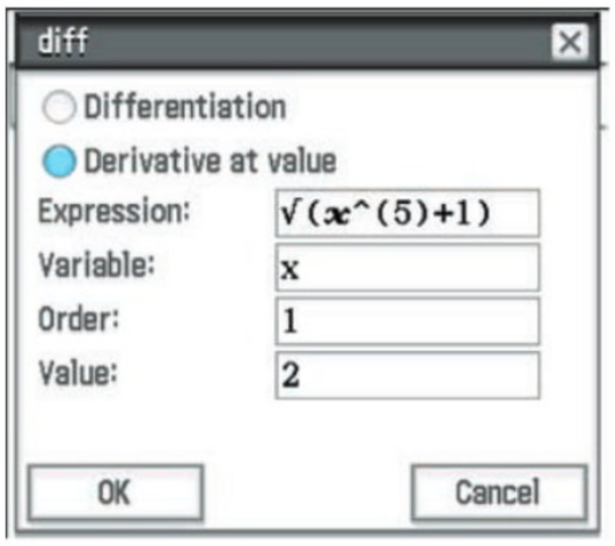
1 Press **menu** > **Calculus** > **Derivative**.
2 In the derivative template, enter variable **x** in the denominator.
3 Enter the expression $\mathbf{x^5}$ in the brackets as shown above.

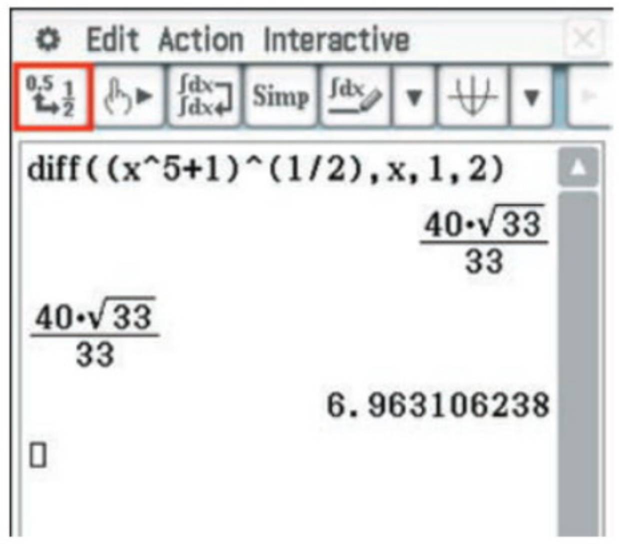
The derivative of x^5 is $5x^4$.

USING CAS 2 Instantaneous rate of change

Determine the instantaneous rate of change at $x = 2$ for the function $f(x) = \sqrt{x^5 + 1}$. State your answer correct to three decimal places.

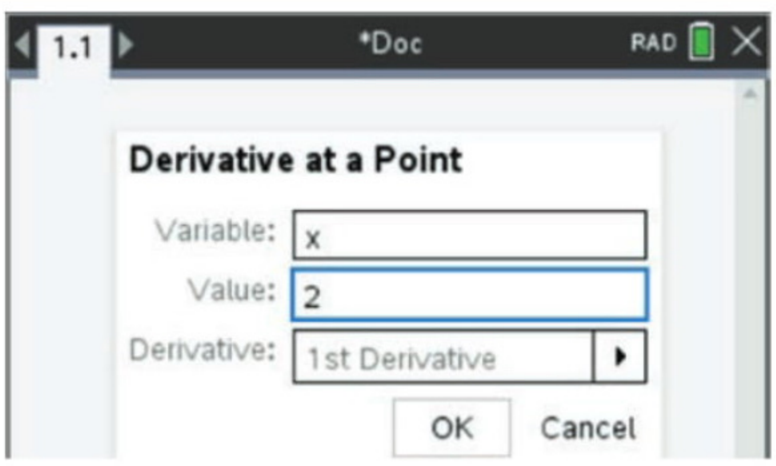
ClassPad

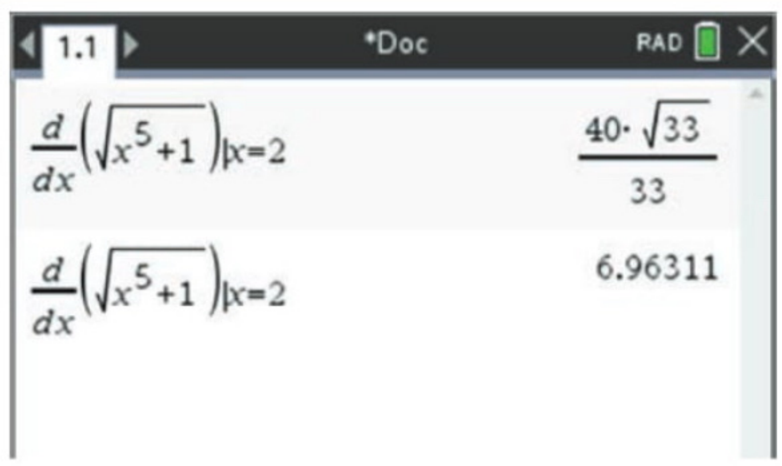




1 In **Main**, enter and highlight the expression $\sqrt{x^5 + 1}$.
2 Tap **Interactive** > **Calculation** > **diff**.
3 In the dialogue box, tap **Derivative at value**.
4 Keep the default **Variable: x** and the **Order: 1**.
5 In the **Value:** field, enter **2**.
6 The exact value of the derivative will be displayed.
7 Change to **Decimal** mode or use the **Convert** tool for the decimal solution.

TI-Nspire





1 Press **menu** > **Calculus** > **Derivative at a Point**.
2 In the dialogue box **Value:** field, enter **2**.
3 Enter the expression inside the brackets.
4 Press **enter** for the exact solution and **ctrl** + **enter** for the approximate solution.

The instantaneous rate of change is 6.963, to three decimal places.

EXERCISE 1.1 Differentiating simple functions

ANSWER p. 387

Mastery

1 WORKED EXAMPLE 1 Differentiate each function.

a $f(x) = -3x^6$

b $f(x) = \dfrac{4}{5x^{\frac{5}{4}}}$

c $f(x) = x^3 + 3\sqrt{5x^{\frac{2}{3}}}$

Exam hack

Write radicals (roots) as powers of x before differentiating, and express answers in terms of a radical if necessary. Write $f(x) = \sqrt{x^5}$ as $f(x) = x^{\frac{5}{2}}$, so $f'(x) = \frac{5}{2}x^{\frac{3}{2}} = \frac{5}{2}\sqrt{x^3}$.

2 WORKED EXAMPLE 2 For each function, write down $f'(x)$ and calculate its exact value using the given value of x.

a $f(x) = 2x^3 - x^2 - 3, f'(-2)$ b $f(x) = \dfrac{x^3 - 1}{x - 1}, f'(1)$ c $f(x) = \dfrac{6x^{\frac{5}{3}} + 9x^{\frac{11}{3}}}{3x^{\frac{2}{3}}}, f'\left(\dfrac{1}{8}\right)$

1.1

3 Using CAS 1 Find the derivative of $f(x) = 5x^3$.

4 Using CAS 2 Given $f(x) = x^2 - \dfrac{1}{3}x^{\frac{3}{2}}$, determine the instantaneous rate of change at $x = 9$.

Calculator-free

5 (3 marks) Let $f(x) = \dfrac{x + x^{\frac{2}{3}} + x^{\frac{3}{4}}}{x^{\frac{1}{2}}}$.

a Write $f'(x)$ in the form $f'(x) = \dfrac{A}{x^{\frac{1}{2}}} + \dfrac{B}{x^{\frac{5}{6}}} + \dfrac{C}{x^{\frac{3}{4}}}$, where A, B and C are constants to be determined. (2 marks)

b Show that $f'(2^{12}) = \dfrac{13}{3 \times 2^9}$. (1 mark)

6 © SCSA MM2021 Q1a (3 marks) Differentiate $\dfrac{3x + 1}{x^3}$ and simplify your answer.

7 (3 marks) Find the value of a, given that $f(x) = \dfrac{1}{a}x^a + a$ and $f'(4) = 16$.

8 (2 marks) Determine the derivative of $f(x) = \left(\sqrt{3x} + \sqrt{5x^3}\right)\left(\sqrt{3x} - \sqrt{5x^3}\right)$.

9 (2 marks) The function $f(x) = \dfrac{1}{3}x^3 + ax^2 + bx + 1$, where a and b are constants, has a turning point at $x = -1$. Given that $f(1) = -2\dfrac{2}{3}$, find the value of a and b.

10 (2 marks) If $f(x) = \dfrac{1}{2x^2}$, obtain expressions for $f'(x + 1)$ and $f'(x - 1)$.

11 (2 marks) Consider the function $f(x) = 4x^3 + 5x - 9$.

a Find $f'(x)$. (1 mark)

b Explain why $f'(x) \geq 5$ for all x. (1 mark)

12 (3 marks) The functions $f(x) = -x(x - a)$ and $g(x) = mx + c$, where a, m and c are constants, have the same gradient at the non-zero x-intercept of $f(x)$. Show that $c = a^2$.

Calculator-assumed

13 (1 mark) Determine y', given $y = 1 - x + \dfrac{1}{3}x^3$.

14 (1 mark) Determine the gradient function of $f(x) = 2\left(\sqrt{x} - \dfrac{1}{3}\sqrt{x^3}\right)$.

1.2 The product rule

Video playlist The product rule

Worksheet The product rule

One way of finding the derivative of $f(x) = (2x + 3)(5x - 1)$ is to first expand brackets to obtain $f(x) = 10x^2 + 13x - 3$ and then differentiate term-by-term to obtain $f'(x) = 20x + 13$.

However, we can also use the **product rule** to differentiate this function because it consists of the product of two functions.

The product rule

If $f(x) = u(x) \times v(x)$, then $f'(x) = u(x) \times v'(x) + v(x) \times u'(x)$.

or

$\frac{d}{dx}(uv)$ is another way of writing the derivative of $f(x) = uv$.

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

This can also be written as $\frac{d}{dx}(uv) = uv' + vu'$.

or

If $y = f(x)g(x)$, then $\frac{dy}{dx} = f(x)g'(x) + g(x)f'(x)$.

To differentiate $f(x) = (2x + 3)(5x - 1)$, let $u = 2x + 3$ and $v = 5x - 1$, and then obtain $u' = 2$ and $v' = 5$.

Now use
$$\begin{aligned}\frac{d}{dx}(uv) &= uv' + vu' \\ &= (2x + 3) \times 5 + (5x - 1) \times 2 \\ &= 20x + 13\end{aligned}$$

or

To differentiate $y = (2x + 3)(5x - 1)$, let $f(x) = 2x + 3$ and $g(x) = 5x - 1$, and then obtain $f'(x) = 2$ and $g'(x) = 5$.

Now use: If $y = f(x)g(x)$, then
$$\begin{aligned}\frac{dy}{dx} &= f(x)g'(x) + g(x)f'(x) \\ &= (2x + 3) \times 5 + (5x - 1) \times 2 \\ &= 20x + 13\end{aligned}$$

Exam hack

The order of the two functions of $f(x)$ does not matter. In the example shown here, we could also have used $u = 5x - 1$ and $v = 2x + 3$.

WORKED EXAMPLE 3 The product rule

Use the product rule to differentiate $y = (5x^3 - 2x)(x^2 + 1)$.

Steps	Working
1 Identify $f(x)$ and $g(x)$.	Let $f(x) = 5x^3 - 2x$ and $g(x) = x^2 + 1$.
2 Differentiate to obtain $f'(x)$ and $g'(x)$.	$f'(x) = 15x^2 - 2, g'(x) = 2x$
3 Write down the expression for $f(x)g'(x) + g(x)f'(x)$.	$y' = (5x^3 - 2x) \times 2x + (x^2 + 1) \times (15x^2 - 2)$
4 Expand and simplify.	$= 10x^4 - 4x^2 + 15x^4 - 2x^2 + 15x^2 - 2$ $= 25x^4 + 9x^2 - 2$

WORKED EXAMPLE 4 The product rule with substitution

For the function $f(x) = (2x^3 - 3x + 1)(5x^4 + x^3 - x + 7)$, find $f'(-3)$.

Steps	Working
1 Identify u and v.	Let $u = 2x^3 - 3x + 1$ and $v = 5x^4 + x^3 - x + 7$.
2 Differentiate to obtain u' and v'.	$u' = 6x^2 - 3$, $v' = 20x^3 + 3x^2 - 1$
3 Write down the expression for $u'v + uv'$.	$f'(x) = (6x^2 - 3)(5x^4 + x^3 - x + 7) + (2x^3 - 3x + 1)(20x^3 + 3x^2 - 1)$
4 Substitute the value and simplify.	$f'(-3) = \left(6(-3)^2 - 3\right)\left(5(-3)^4 + (-3)^3 - (-3) + 7\right) + \left((2(-3)^3 - 3(-3) + 1)(20(-3)^3 + 3(-3)^2 - 1)\right)$ $= 51 \times 388 - 44 \times (-514)$ $= 42\,404$

Exam hack

It is usually easier and faster to substitute the value into the long expression than to expand and simplify the long expression and then substitute.

WORKED EXAMPLE 5 Tangents and stationary points

For the function $f(x) = (x - 1)(2x + 3)$, determine

a the equation of the tangent to the curve at $x = 2$

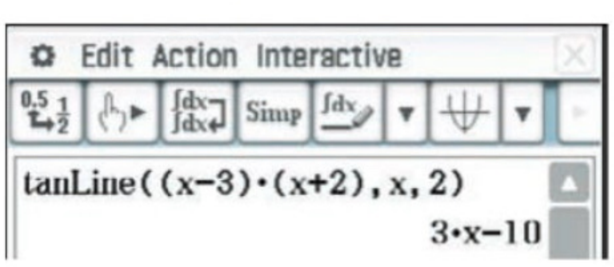
b the coordinates of any stationary points.

Steps	Working
a 1 Differentiate $f(x)$ using the product rule.	$f'(x) = (2x + 3) + 2(x - 1) = 4x + 1$
2 Determine $f(2)$ and $f'(2)$.	$f(2) = 7, f'(2) = 4(2) + 1 = 9$
3 Determine the equation of the tangent.	Using $y = mx + c, 7 = 9 \times 2 + c \Rightarrow c = -11$ $y = 9x - 11$
b 1 Let $f'(x) = 0$ and solve for x.	$4x + 1 = 0 \therefore x = -\frac{1}{4}$
2 Determine $f\left(-\frac{1}{4}\right)$.	$f\left(-\frac{1}{4}\right) = -\frac{25}{8}$
3 State the coordinates of stationary point.	$\left(-\frac{1}{4}, -\frac{25}{8}\right)$

USING CAS 3 Finding the equations of tangent lines

For the function $y = (x - 3)(x + 2)$, determine the equation of the tangent to the curve at $x = 2$.

ClassPad



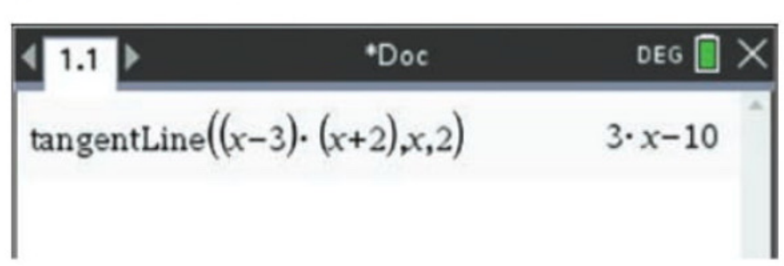
1 Enter and highlight the expression **(x – 3)(x + 2)**.

2 Tap **Interactive > Calculation > line > tanLine**.

3 In the dialogue box, set the **Point** field to 2 and tap **OK**.

4 The expression for the tangent line will be displayed.

TI-Nspire



1 Press **menu > Calculus > tangent Line**.

2 Enter **(x – 3)(x + 2),x,2** and press **enter**.

3 The expression for the tangent line will be displayed.

The tangent line is $y = 3x - 10$.

EXERCISE 1.2 The product rule

ANSWERS p. 387

Recap

1 If $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2$, then $f'(a)$ is

A a^2 **B** $a^2(a+1)$ **C** $1 - a^2$ **D** $a(a-1)$ **E** $a+1$

2 For the function $f(x) = \sqrt{x} + \frac{1}{\sqrt{x}}$, $f'(x)$ is

A $\frac{\sqrt{x}}{2}\left(1 + \frac{1}{x}\right)$ **B** $\frac{1}{2\sqrt{x}}\left(1 - \frac{1}{x}\right)$ **C** $\frac{1}{2}\left(1 - \frac{1}{\sqrt{x}}\right)$ **D** $\frac{1}{\sqrt{x}}(1 + \sqrt{x})$ **E** $\frac{2}{\sqrt{x}}(x-1)$

Mastery

3 WORKED EXAMPLE 3 Use the product rule to differentiate $f(x) = (4x + 3x^2)(7x^2 - 1)$.

4 Differentiate each expression and simplify.

a $x^4(3x+1)$ **b** $(4x+3)(3x-2)$ **c** $7x(8x-5)$

d $-x^5(4-x^2)$ **e** $4x(x^5 - x^2)$ **f** $(5x-7)(5x+7)$

g $(1+3x)(x^2-1)$ **h** $(4x+5)(2x^3 - 2x + 1)$ **i** $(x^2+1)^2$

5 WORKED EXAMPLE 4 Find $f'(-1)$ for the function $f(x) = (1 + 3x - 2x^3)(6 + x^3 - x^5)$.

6 **a** Differentiate $f(x) = (x^4+1)(2x^3+5) + (3x^2-4)(2x^2+5)$.

b Evaluate $f'(-1)$.

7 If $f(x) = (x^2 + x + 2)(x - 3)$, determine $f'(3)$.

8 WORKED EXAMPLE 5 For the function $f(x) = (2x-1)(x-3)$, determine

a the equation of the tangent to the curve at $x = 2$

b the coordinates of any stationary points.

9 Using CAS 3 For the function $y = (x+3)(x-2)$, determine the equation of the tangent to the curve at $x = -2$.

Calculator-free

10 (1 mark) Determine the gradient of the curve $y = \sqrt{x}(x^2+1)$ at $x = 4$.

11 (2 marks) Show that for the functions $f(x) = a + bx^2$ and $g(x) = c + dx^2$, where a, b, c and d are constants, if $f'(x)g(x) = f(x)g'(x)$, then $bc = ad$.

12 (3 marks) Let $y = (x-a)^2(x-b)$.
Determine the values of a and b when $\frac{dy}{dx} = (x-5)(3x-11)$.

13 (3 marks) Given that $f'(6) = 6$ for the function $f(x) = (ax-4)(ax+3)$, where a is a constant, use the product rule to determine the possible values of a.

14 (4 marks) Let $f(x) = (ax+b)(bx+a)$ for positive constants a, b.

a Use the product rule to find $f'(x)$. (2 marks)

b Determine values for a and b when $f'(1) = 25$ and $f'(2) = 37$. (2 marks)

Calculator-assumed

15 (2 marks) Find the coordinates of the turning points on the curve $y = (x-6)(x^2-9)$.

16 (1 mark) Determine the gradient of the tangent to the curve $y = \sqrt[3]{\frac{1}{x^2}}(1-x^2)$ at $x = 2$, correct to one decimal place.

17 (2 marks) Determine the coordinates of the point on the curve $y = (3-x)\sqrt{x^3}$ where the gradient is zero.

18 (2 marks) Determine the value of a if $\frac{d}{dx}[x^2(2-ax^3)] = 4x - \frac{5x^4}{2}$.

1.3 The quotient rule

Video playlist The quotient rule

Worksheet The quotient rule

We use the **quotient rule** to differentiate a function consisting of the ratio of two functions.

The quotient rule

If $f(x) = \frac{u(x)}{v(x)}$,

then $f'(x) = \frac{v(x) \times u'(x) - u(x) \times v'(x)}{(v(x))^2}$ or $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ or $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$.

or

If $y = \frac{f(x)}{g(x)}$, then $\frac{dy}{dx} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$.

Exam hack

When using the quotient rule, differentiate in alphabetical order: u' first, then v' or $f'(x)$ first then $g'(x)$.

WORKED EXAMPLE 6 The quotient rule

Find the derivative of $\frac{8x-3}{4x+5}$.

Steps	Working
1 Identify u and v.	$u = 8x - 3$, $v = 4x + 5$
2 Differentiate to obtain u' and v'.	$u' = 8$, $v' = 4$
3 Write down the expression for $\frac{vu' - uv'}{v^2}$.	$\frac{d}{dx}\left(\frac{8x-3}{4x+5}\right) = \frac{(4x+5) \times 8 - (8x-3) \times 4}{(4x+5)^2}$
4 Expand and simplify.	$= \frac{32x + 40 - 32x + 12}{(4x+5)^2}$ $= \frac{52}{(4x+5)^2}$

WORKED EXAMPLE 7 The quotient rule with substitution

If $y = \dfrac{x^2 + x}{-2x + 3}$, determine $\dfrac{dy}{dx}$ at $x = 3$.

Steps	Working
1 Identify $f(x)$ and $g(x)$.	$f(x) = x^2 + x, g(x) = -2x + 3$
2 Differentiate to obtain $f'(x)$ and $g'(x)$.	$f'(x) = 2x + 1, g'(x) = -2$
3 Write down the expression for $\dfrac{dy}{dx}$.	$\dfrac{dy}{dx} = \dfrac{(-2x + 3)(2x + 1) - (x^2 + x)(-2)}{(-2x + 3)^2}$
4 Evaluate $\dfrac{dy}{dx}$ at $x = 3$.	$\dfrac{dy}{dx} = \dfrac{(-2(3) + 3)(2(3) + 1) - ((3)^2 + 3)(-2)}{(-2(3) + 3)^2}$ $= \dfrac{1}{3}$

EXERCISE 1.3 The quotient rule

ANSWERS p. 387

Recap

1 The value of $f'(4a^2)$ for the function $f(x) = x(\sqrt{x} - 1)$ is

A $a^2 - 1$ **B** $3a + 3$ **C** $a + 1$ **D** $a^3 + 3$ **E** $3a - 1$

2 Determine the gradient of the function $g(x) = (x + 3)(2x - 5)$ at $x = 2$.

Mastery

3 WORKED EXAMPLE 6 Find the derivative of $\dfrac{6x - 1}{9x - 8}$.

4 Find the derivative of each of the following.

a $\dfrac{2}{2x + 3}$ **b** $\dfrac{4 - x}{x - 5}$ **c** $\dfrac{x - 1}{x + 1}$ **d** $\dfrac{1}{x(x + 1)}$

e $\dfrac{x^3 + x}{x + 3}$ **f** $\dfrac{1 + x + x^2}{x}$ **g** $\dfrac{x^3 - 1}{x - 1}$ **h** $\dfrac{3(2x + 5)}{1 - x^2}$

5 WORKED EXAMPLE 7 If $y = \dfrac{2x - 1}{x + 4}$, determine $\dfrac{dy}{dx}$ at $x = 2$.

Calculator-free

6 (4 marks) Determine the value of the integer constant k in the function $f(x) = \dfrac{x + k}{x - k}$ given that $f'(5) = -8$.

7 (1 mark) Let $f(x) = \dfrac{1}{2x - 4} + 3$. Find $f'(x)$.

8 (2 marks) If $f(x) = \dfrac{2 - x + x^2}{x^2 - 2x + 1}$, determine $f'(2)$.

9 (5 marks) Determine the equation of the tangent to the curve $y = \frac{2+x}{2-x}$ at $x = 4$. Also, determine the coordinates of the x and y intercepts of this tangent.

1.4

10 (4 marks) For the function $f(x) = \frac{5}{2x-1}$, find the values of the constants a and b if $(2x-1)f'(x) = -\frac{a}{x+b}$.

11 (3 marks) If $y = \frac{1}{1+x}$, show that $\frac{dy}{dx} = -y^2$.

Calculator-assumed

12 (1 mark) If $y = \frac{x^2}{x+2}$, then determine an expression for $\frac{dy}{dx}$.

13 (2 marks) Given $f(x) = \frac{x^2}{x+1}$ and $g(x) = \frac{3x+2}{x+2}$, find all values of x that satisfy $f'(x) = g'(x)$. Give answers to two decimal places.

14 (3 marks) Determine the exact gradient of the curve $y = \frac{x^2-4}{x^2-3x-4}$ at the points where the curve intersects the coordinate axes.

15 (3 marks) If the derivative of $y = \frac{(x+2)^2}{x^2+a}$ for a unique value of a is $\frac{8-28x-16x^2}{(2x^2+1)^2}$, then determine the value of a.

16 (3 marks) Find the values of a so that the function $f(x) = ax + \frac{x^2+1}{x+1}$ has at least one stationary point.

1.4 The chain rule

Video playlist The chain rule

Worksheets The chain rule

Mixed differentiation problems

If $f(x) = x^2 + 2$ and $g(x) = x^3$, then the **composition of functions** f and g, is $f(g(x)) = (x^3)^2 + 2$.

The **chain rule** transforms a **composite function** of the form $y = f(g(x))$ to make differentiation easier or, in some cases, possible.

For example, $y = \sqrt{5x-4}$ is not a standard function, but by letting $u = 5x - 4$, we get $y = \sqrt{u} = u^{\frac{1}{2}}$.

This is differentiated with respect to the variable u to obtain $\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$.

Then we differentiate $u = 5x - 4$ and apply the chain rule.

The chain rule

If $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

or

If $y = f(g(x))$, then $\frac{dy}{dx} = f'(g(x))g'(x)$.

WORKED EXAMPLE 8 The chain rule

Differentiate each function with respect to x using the chain rule.

a $\dfrac{1}{(3x+1)^4}$

b $\sqrt{5x-4}$

Steps	Working
a 1 Write $\dfrac{1}{(3x+1)^4}$ as a function of a function in index form.	Let $u = 3x + 1$. Then $y = \dfrac{1}{(3x+1)^4} = (3x+1)^{-4} = u^{-4}$
2 Write the chain rule and find the two derivatives.	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$ $\dfrac{dy}{du} = -4u^{-5}, \dfrac{du}{dx} = 3$
3 Substitute the derivatives.	$\dfrac{dy}{dx} = -4u^{-5} \times 3$
4 Substitute for u.	$= -12(3x+1)^{-5}$
5 Write the answer.	$\dfrac{d}{dx}\left[\dfrac{1}{(3x+1)^4}\right] = -\dfrac{12}{(3x+1)^5}$
b 1 Determine $f(x)$ and $g(x)$.	$f(x) = \sqrt{x}$, $g(x) = 5x - 4$.
2 Write the chain rule.	$\dfrac{dy}{dx} = f'(g(x))g'(x)$
3 Differentiate.	$\dfrac{dy}{dx} = \dfrac{1}{2}(5x-4)^{-\frac{1}{2}}(5)$
4 Simpify.	$\dfrac{dy}{dx} = \dfrac{5}{2\sqrt{5x-4}}$

EXERCISE 1.4 The chain rule

ANSWERS p. 388

Recap

1 Determine $f'(x)$ for the function $f(x) = \dfrac{x^2}{2x-3}$.

2 The gradient of the tangent to the function $f(x) = \dfrac{\sqrt[3]{x}}{x+2}$ at $x = 27$ is

A $-\dfrac{52}{22707}$ **B** $\dfrac{26}{22707}$ **C** $\dfrac{52}{7569}$ **D** $\dfrac{26}{2523}$ **E** $\dfrac{52}{783}$

Mastery

3 WORKED EXAMPLE 8 Differentiate each expression with respect to x using the chain rule.

a $\dfrac{2}{(x^3+1)^4}$

b $\sqrt{x^2-1}$

4 Differentiate each of the following with respect to x.

a $(x-5)^5$ **b** $(4x-3)^4$ **c** $(2x^3+x)^3$ **d** $(8-2x^2)^6$

e $\left(\frac{1}{2}x-6\right)^9$ **f** $(x^3-2x^2+x+1)^2$ **g** $(4x+6)^{\frac{1}{2}}$ **h** $(2\sqrt{x}-x)^3$

i $\sqrt{5(x+10)}$ **j** $\frac{1}{(2x+7)^2}$ **k** $\frac{1}{\sqrt{4-x}}$ **l** $\frac{5}{\sqrt{(x-8)^3}}$

5 Find the derivative of each expression.

a $(2x-1)^4$ **b** $(3-x^3)^2$

c $(3+4x+2x^2)^7$ **d** $(x^2+6x)^6$

e $(x^3-x^6+1)^5$ **f** $\frac{1}{n+1}(x^{n+1}+1)^{n+1}$ for positive integers, n

Calculator-free

6 (2 marks) Let $y=(3x^2-5x)^5$. Find $\frac{dy}{dx}$.

7 (1 mark) If $y=(x^2-5x)^4$, find $\frac{dy}{dx}$.

8 (1 mark) Differentiate $\sqrt{4-x}$ with respect to x.

9 (1 mark) If $y=(-3x^3+x^2-64)^3$, find $\frac{dy}{dx}$.

10 (2 marks) Given $f(x)=(x^2+ax+1)^3$, state the value of a if $f'(0)=3$.

11 (2 marks) Show that if $x>a$, the gradient at any point on the graph of the function $y=\sqrt{1+(x-a)^2}$ is positive.

12 (2 marks) Show that if $y=\sqrt{1-f(x)}$, $\frac{dy}{dx}$ is equal to $\frac{-f'(x)}{2\sqrt{1-f(x)}}$.

13 (2 marks) Show that if $f(x)=(x-a)^2g(x)$, then the derivative of $f(x)$ is $(x-a)[2g(x)+(x-a)g'(x)]$.

Calculator-assumed

14 (2 marks) Determine the exact value of the instantaneous rate of change in w when $v=3$, given that $w=\frac{3}{\sqrt[3]{9+2v^2}}$.

15 (3 marks) If $f(x)=a(bx+1)^3$, determine the values of a and b given that $f(0)=2$ and $f'(0)=18$.

16 (3 marks) Determine the values of a and b when $f'[g(x)]=\sqrt{a(6x-7)^b}$, $f(x)=\sqrt{x^3}$, and $g(x)=6x-7$.

17 (3 marks) The height, h cm, of a tomato plant is a function of the amount of compost, c grams, it receives. The function is $h(c)=c^2+c+1$, where $c(t)=t^3+t$, with t metres being the thickness of the topsoil. Express the rate of change of the height of the tomato plant with respect to topsoil thickness in the form $\frac{dh}{dt}=f(t)$ and calculate $\frac{dh}{dt}$ when $t=10$ cm, correct to one decimal place.

Video playlist
Combining the rules

1.5 Combining the rules

Sometimes a combination of the product, quotient and chain rules is required. Each question needs to be carefully considered to see which method, or combination of methods, is appropriate.

There may be more than one way to complete a question.

WORKED EXAMPLE 9 Combining the chain and product rules

Differentiate $2x^5(5x+3)^3$.

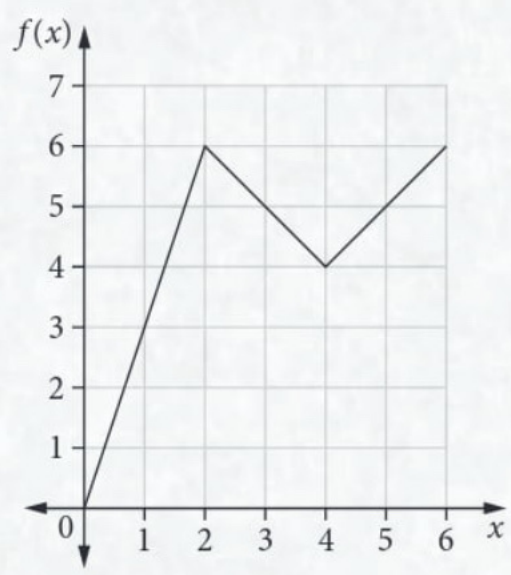
Steps	Working
1 Write as a product.	Let $y = uv$, where $u = 2x^5$ and $v = (5x+3)^3$.
2 Find the derivative of u.	$u' = 10x^4$
3 Write v as a function of a function.	Let $v(x) = p(q(x))$, where $p(q) = q^3$ and $q(x) = 5x + 3$.
4 Write the chain rule.	$v' = \frac{dp}{dx} = \frac{dp}{dq} \times \frac{dq}{dx}$
5 Substitute the derivatives.	$= 3q^2 \times 5$
6 Substitute for q.	$= 15(5x+3)^2$
7 Write the product rule.	$y = uv' + vu'$
8 Substitute the functions.	$= 2x^5 \times 15(5x+3)^2 + (5x+3)^3 10x^4$ $= 30x^5(5x+3)^2 + 10x^4(5x+3)^3$
9 Take out the common factor.	$= 10x^4(5x+3)^2[3x + (5x+3)]$
10 Simplify and write the answer (optional).	$= 10x^4(5x+3)^2(8x+3)$

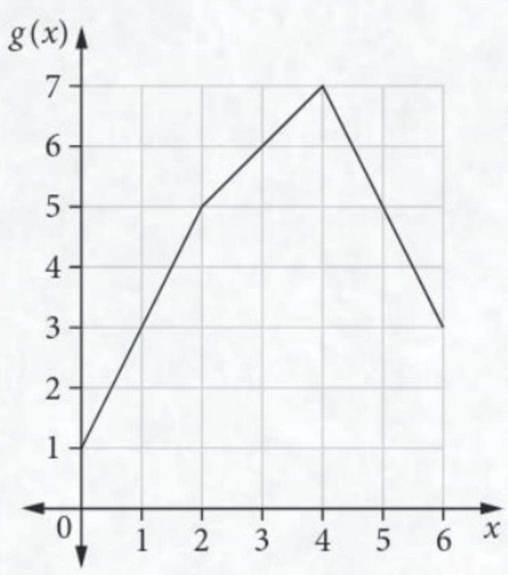
Video
WACE question analysis: Differentiation

WACE QUESTION ANALYSIS

© SCSA MM2020 Q5 Calculator-free (5 marks)

The graphs of the functions f and g are displayed below.





a Evaluate the derivative of $f(x)$ at $x = 3$. (1 mark)

b Evaluate the derivative of $f(x)g(x)$ at $x = 5$. (2 marks)

c Evaluate the derivative of $f(g(x))$ at $x = 1$. (2 marks)

Reading the question

- Understand key words and terms because they provide a good indication of the approach to take. For instance, evaluate means you should provide a numerical value for your answer.
- Note what is asked in each question. For instance, each part is looking for an answer at a particular point. Consider how this fits with the information given in the body of the question and the graphs displayed.
- Take note of the ways the functions are written in parts **b** and **c**. Does this look familiar in terms of the rules you have learnt – product, quotient and/or chain rules?

Thinking about the question

- As parts **b** and **c** are only worth two marks, no working or explanation is required to gain full marks. However, if your final answer is wrong, it may still be possible to obtain part marks for working.
- Note that the graphs relate to functions, but the questions relate to derivatives. Consider how the graphs can give you information on the gradients, and hence the derivatives.
- Remember to answer each part of the question fully.

Worked solution (✓ = 1 mark)

a $f'(3) = -1$

states the correct derivative ✓

b $(fg)'(5) = f'(5)g(5) + g'(5)f(5)$

$= (1)(5) + (-2)(5)$

$= -5$

uses product rule to express derivative ✓

states correct derivative ✓

c $f(g(x)')$ when $x = 1 = f'(g(1))g'(1)$

$= f'(3)2$

$= (-1)2$

$= -2$

uses chain rule to express derivative ✓

states correct derivative ✓

EXERCISE 1.5 Combining the rules

ANSWERS p. 388

Recap

1 The derivative of $(1 - 0.5x^4)^4$ at $x = 1$ is

A -1 **B** $-\frac{1}{2}$ **C** 0 **D** $\frac{1}{4}$ **E** 1

2 $\frac{d}{dx}\left(\sqrt{(x^2+1)^3}\right)$ is

A $\frac{3}{x^2+1}$ **B** $\frac{\sqrt{3x}}{(x^2+1)^2}$ **C** $\frac{3x}{x^2+1}$ **D** $\frac{1}{2\sqrt{x^2+1}}$ **E** $3x\sqrt{x^2+1}$

Mastery

3 WORKED EXAMPLE 9 Differentiate the function $x^2(2x+1)^3$.

4 Consider the function $f(x) = (x-1)^2(x-2) + 1$.

If $f'(x) = (x-1)(ux+v)$, where u and v are constants, use calculus to find the values of u and v.

5 Determine the derivative of $10p(1-p)^9$ with respect to p.

Calculator-free

6 (6 marks)

a Differentiate $\dfrac{x+1}{2x-1}$

i using the quotient rule (2 marks)

ii by expressing the function as a product and using the product rule. (2 marks)

b Show that your answers in part **a** are the same. (2 marks)

7 (4 marks) The gradient of the function $f(x) = \dfrac{x+a}{x-a}$ (a is an integer) at $x = 1$ is $-\dfrac{3}{2}$. Find the value of a.

8 (4 marks) If $y = \dfrac{x}{\sqrt{a^2 - x^2}}$, show that $\dfrac{dy}{dx} = \dfrac{a^2}{\sqrt{(a^2 - x^2)^3}}$.

9 (3 marks) If $y = \sqrt{x^2 + 3}$ and $x(t) = 4t^3 + t + 1$, evaluate the rate of change of y with respect to t when $t = 0$.

10 (3 marks) Differentiate $f(x) = \dfrac{ax^2 + b}{x + b}$ and determine the values of a and b given that they are integers, and $f(1) = 1$ and $f'(-1) = -2$.

11 (3 marks) Let $f(x) = mx$ where $m \neq 0$, and $g(x) = (f(x))^n$ for positive integers n. Show that $g'(x) = \dfrac{ng(x)}{x}$.

Calculator-assumed

12 (4 marks)

a Differentiate each of the functions $f(x) = (x^2 - x + 1)^2$ and $g(x) = (x + a)^3$. (2 marks)

b Show that at $x = 0$ the tangents to $y = f(x)$ and $y = g(x)$ are perpendicular if $6a^2 = 1$. (2 marks)

13 (3 marks) If $y = \dfrac{x^2}{1 + x^2}$, show that $(1 + x^2)\dfrac{dy}{dx} + 2xy = 2x$.

14 (4 marks)

a Give the values of a and b so that the derivative of the function $f(x) = \dfrac{(6 - x^2)x}{3(2 - x^2)}$ is in the form $\dfrac{ax^4 + b}{3(x^2 - 2)^2}$. (2 marks)

b Determine the coordinates of the point on the curve of $f(x)$ for $0 \le x \le 5$, where $f(x) = f'(x)$. Give the answer correct to three decimal places. (2 marks)

15 (4 marks) The point $(2, b)$ lies on the graph of the function $y = \dfrac{a + 4x^2}{2x + 1}$. The gradient of the curve at that point is $\dfrac{6}{5}$. Determine the value of a and b.

1 Chapter summary

Differentiation

- To differentiate means to find the instantaneous rate of change at a given point.
  - **Instantaneous rate of change** at point A is the gradient of the **tangent** at a point A.
  - The **tangent** to a curve is a straight line touching the curve at a point.
  - A **stationary point** is a point on a curve where the gradient is zero (that is, where $f'(x) = 0$).

Derivative of a power of x

- If $f(x) = ax^n$, then $f'(x) = anx^{n-1}$.

The product rule

- The product rule is used to differentiate the product of two functions $u = u(x)$ and $v = v(x)$.

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \quad \text{or} \quad \frac{d}{dx}(uv) = uv' + vu'.$$

or

If $y = f(x)g(x)$, then $\frac{dy}{dx} = f(x)g'(x) + g(x)f'(x)$.

The quotient rule

- The quotient rule is used to differentiate the ratio of two functions $u = u(x)$ and $v = v(x)$.

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \quad \text{or} \quad \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

or

If $y = \frac{f(x)}{g(x)}$ then $\frac{dy}{dx} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$.

The chain rule

- The chain rule differentiates a composite function $y = f(g(x))$.

If $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

or

If $y = f(g(x))$, then $\frac{dy}{dx} = f'(g(x))g'(x)$.

Cumulative examination: Calculator-free

Total number of marks: 16 Reading time: 2 minutes Working time: 16 minutes

1 (1 mark) Let $y = (5x + 1)^7$. Find $\frac{dy}{dx}$.

2 © SCSA MM2018 Q3a (2 marks) Differentiate $(2x^3 + 1)^5$.

3 (4 marks)

a Let $f(x) = \frac{x}{x + 2}$.

Differentiate f with respect to x. (2 marks)

b Let $g(x) = (2 - x^3)^3$.

Evaluate $g'(1)$. (2 marks)

4 (1 mark) Let $f(x) = \sqrt{1 - 2x}$. Find $f'(x)$.

5 (3 marks) Consider the tangent to the graph of $y = x^2$ at the point $(2, 4)$. Show that the point $(3, 8)$ lies on this tangent.

6 (1 mark) Let $f(x) = \frac{1}{5}(x - 2)^2(5 - x)$.

Write down the derivative $f'(x)$.

7 © SCSA MM2019 Q2ab (4 marks) The values of the functions $g(x)$ and $h(x)$, and their derivatives $g'(x)$ and $h'(x)$ are provided in the table below for $x = 1$, $x = 2$ and $x = 3$.

	$x = 1$	$x = 2$	$x = 3$
$g(x)$	3	5	−3
$h(x)$	2	−2	6
$g'(x)$	−4	1	4
$h'(x)$	0	−6	−5

a Evaluate the derivative of $\frac{g(x)}{h(x)}$ at $x = 3$. (2 marks)

b Evaluate the derivative of $h(g(x))$ at $x = 1$. (2 marks)

Cumulative examination: Calculator-assumed

Total number of marks: 14 Reading time: 2 minutes Working time: 14 minutes

1 (2 marks) Determine the equation of the tangent to the curve $f(x) = (3x + 2)(x^4 + x^3)$ at the point where $x = 1$.

2 (4 marks) Given $y = \frac{(5 - x)^3}{\sqrt{2x + 1}}$, show that $\frac{dy}{dx} = \frac{-(5 - x)^2(5x + 8)}{(2x + 1)^{\frac{3}{2}}}$.

3 (4 marks) The point $(1, -3)$ lies on the curve $y = \frac{a + bx}{2x - 5}$ and the gradient at that point is $-\frac{11}{3}$.
Determine the value of a and b.

4 (4 marks) Using **calculus techniques** determine the coordinates of all stationary points for the function $y = \frac{1}{3}x^3 + 2x^2 + 3x - 2$.